Tangent circles in the ratio $2 : 1$

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In this article we consider the following old Japanese geometry problem (see Figure 1), whose statement in [1, p. 39] is missing the condition that two of the vertices are the opposite ends of a diameter. (The authors implicitly correct the omission in the proof they provide on page 118.) We denote by $O(r)$ the circle with centre O, radius r.

Problem [1, Example 3.2]. The squares $ACB'D'$ and $ABC'D$ have a common vertex A, and the vertices C and B', C' and B lie on the circle $O(R)$ whose diameter is $B'C'$, A lying within the circle. The circle $O_1(r_1)$ touches AB and AC and also internally touches $O(R)$, and $O_2(r_2)$ is the incircle of triangle ABC. Show that

$$r_1 = 2r_2 . \tag{1}$$

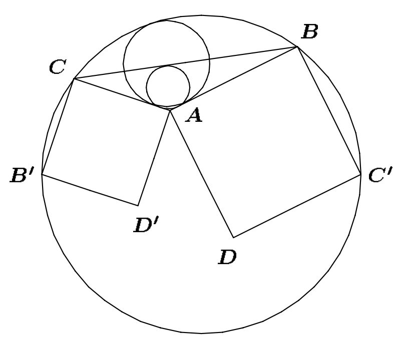

Figure 1.

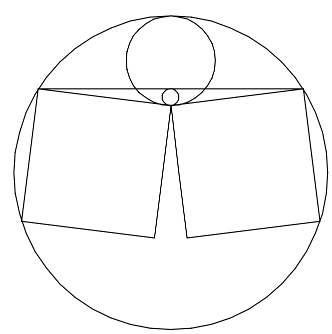

Figure 2.

We shall see that $B'C'$, being a diameter, is merely a sufficient condition. Figure 2 shows that some additional property involving B' and C' is required for deducing (1). In Theorem 1 we give a simple condition that implies (1).

Theorem 1. For any triangle ABC, if A' is the reflection in BC of a point A, then the radius of one of the circles internally touching the circumcircle of $A'BC$ and also touching AB and AC is twice the size of the radius of the incircle of ABC.

Proof. Let AA' intersect BC at P and the circumcircle γ (say) of $A'BC$ again at D, and let Q be the foot of the perpendicular from C to BA' (see Figure 3). Then $\angle BA'D = \angle QCB$ since the right triangles $BA'P$ and BCQ share a common angle $\angle A'BC$. Moreover $\angle BA'D = \angle BCD$. Hence, $\angle QCB = \angle BCD$. This implies that H and D are symmetric in BC, where H is the orthocentre of $A'BC$. Thus, D is the orthocentre of ABC, and therefore, DBC and ABC share a common nine-point circle β (say), and β touches the incircle α (say) of ABC internally by Feuerbach's Theorem. Therefore, the dilatation of magnification 2 with centre A carries β into the circumcircle of DBC, which is γ, and α into one of the circles touching AB, AC and γ internally. This implies that the last circle is twice the size of α, and the proof is complete.

We mentioned "one of the circles" in the theorem, but if we introduce orientations of lines and circles, the circle is determined uniquely. Let us assume that γ and ABC have counter-clockwise orientations. Then the circle of twice the size of α (illustrated by a dotted line in Figure 3) is the one which touches γ, $\overrightarrow{AB}$ and $\overrightarrow{CA}$ so that the orientations at the points of tangency are the same (see the arrows in Figure 3), and the point of tangency of the circle with γ is the reflection of A in the point of tangency of α and β.

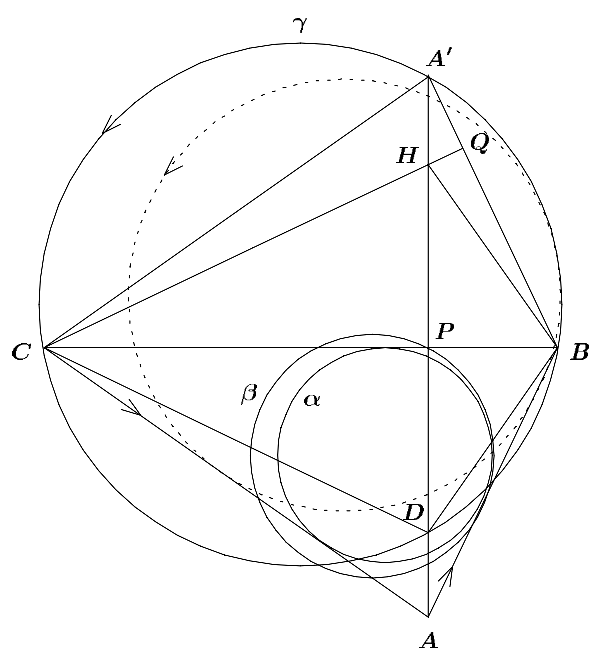

Figure 3.

The following two properties follow immediately (see Figures 4 and 5). They can be found without a proof in [1, p. 29].

Corollary 1. If ABC is a right triangle with right angle at A, then the circle touching the circumcircle of ABC internally and also touching AB and AC is twice the size of the incircle of ABC.

Corollary 2. If $CB'A$ is an isosceles triangle with $CB' = CA$, and B is a point lying on the line $B'A$, then one of the circles touching the circumcircle of BCB' internally and also touching AB and AC is twice the size of the incircle of ABC.

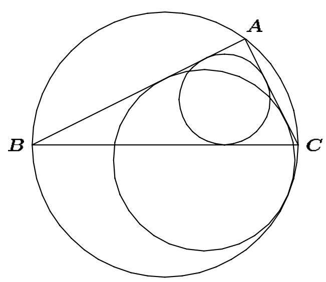

Figure 4.

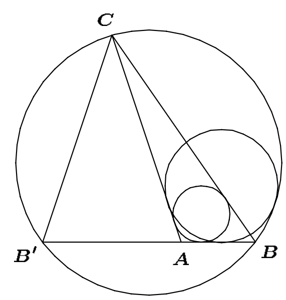

Figure 5.

The last corollary holds since the reflection in BC of A lies on the circumcircle of BCB'. Though Figure 5 illustrates only the case where A lies on the segment BB', and this is the case stated in [1], the corollary does not need this condition (see Figure 6b).

Conversely, for a triangle ABC, let A' be the reflection in BC of A, and let B' be the intersection of the line AB and the circumcircle of $A'BC$. Then $CA = CB'$ holds. Hence, with the two sides AB and AC and their intersections with the circumcircle, we can construct two similar isosceles triangles (see Figures 6a and 6b, where the ratio of the two smaller circles is $2 : 1$). A lies within the circumcircle if and only if $\angle CAB > 90°$, and Figure 1 can be obtained by letting $\angle CAB = 135°$.

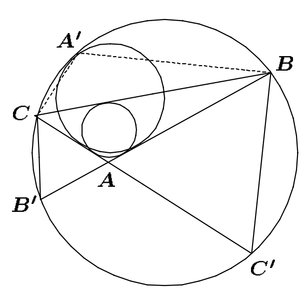

Figure 6a.

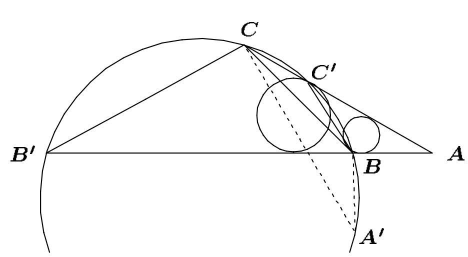

Figure 6b.

The following property with a proof using trigonometric functions can be found in [2, p. 75] (see Figure 7).

Corollary 3. If $ACB'DE$ and $ABC'D'E$ are two regular pentagons sharing the side AE, then the circle internally touching the circumscribed circle of $CB'C'B$ and the sides AB and AC is twice the size of the incircle of DED'.

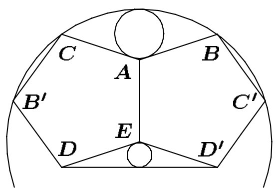

Figure 7.

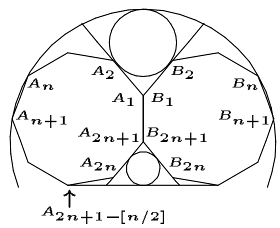

Figure 8.

The corollary can be generalized yet further (see Figure 8).

Theorem 2. If A_1, A_2, $\cdots$, A_{2n+1} are vertices of a regular $(2n+1)$–gon lying in this order, and B_i is the reflection of A_i in the line A_1A_{2n+1}, and γ is the circle passing through A_n, A_{n+1}, B_{n+1}, B_n, then one of the circles touching the lines A_1A_2, B_1B_2 and γ internally is twice the size of the incircle of the triangle made by the lines $A_{2n}A_{2n+1}$, $B_{2n}B_{2n+1}$ and $A_{2n+1-[n/2]}B_{2n+1-[n/2]}$, where $[x]$ is the largest integer which does not exceed x.

Proof. A_{2n+1} is the centre of γ, and the reflection of $A_{2n+1-[n/2]}$ in the line through the centres of the two regular polygons is $A_{1+[n/2]}$. Let us produce $A_{2n+1}A_1$ to P, where P lies on γ and A_1 lies on the segment $A_{2n+1}P$, and let Q be the intersection of A_1P and $A_{1+[n/2]}B_{1+[n/2]}$. By simple calculation we have

$$A_{2n+1}P = 2r\cos\frac{\pi}{2(2n+1)}, \qquad A_{2n+1}A_1 = 2r\sin\frac{\pi}{2n+1},$$

where r is the circumradius of $A_1A_2\cdots A_{2n+1}$. Now let us suppose that n is odd; then $[n/2]=(n-1)/2$ and we have

$$\begin{aligned}
A_1Q &= r\sin\Big(\frac{2\pi}{2n+1}\cdot\frac{n-1}{2}+\frac{\pi}{2n+1}\Big)-\frac{A_{2n+1}A_1}{2}\\
&= r\Big(\sin\frac{n}{2n+1}\pi-\sin\frac{1}{2n+1}\pi\Big),\\
A_{2n+1}Q &= A_{2n+1}A_1+A_1Q\\
&= r\Big(\sin\frac{n}{2n+1}\pi+\sin\frac{1}{2n+1}\pi\Big),
\end{aligned}$$

$$\begin{aligned} A_1Q + A_{2n+1}Q &= 2r \sin \frac{n}{2n+1}\pi = 2r \sin \left(\frac{1}{2}\pi - \frac{1}{2(2n+1)}\pi\right) \\ &= 2r \cos \frac{1}{2(2n+1)}\pi \, . \end{aligned}$$

Therefore, we get $A_1Q + A_{2n+1}Q = A_{2n+1}P$, and this implies that Q is the mid-point of A_1P. Similarly, we can prove the same fact in the case of n being even. Thus, the ratio of the two similar isosceles triangles made by the lines A_1A_2, B_1B_2, $A_{1+[n/2]}B_{1+[n/2]}$, and A_1A_2, B_1B_2, the tangent of γ at P, is $1 : 2$ and the theorem is proved.

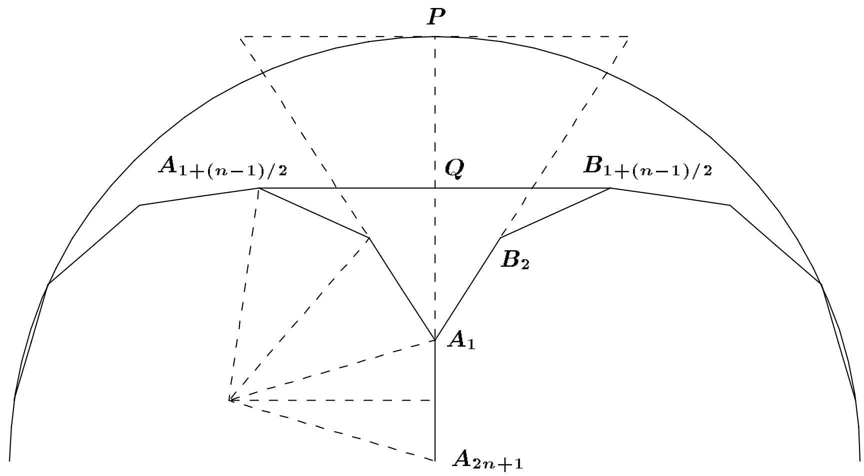

Figure 9 (when n is odd).

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References

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